# Exercise 12

In Exercises 9–12, show that the given function $u(x)$ is a solution of the corresponding Fredholm integro-differential equation:

$$u'''(x) = 1 + \sin x - \int_0^{\frac{\pi}{2}} (x - t)u(t)\,dt,\ u(0) = 1,\ u'(0) = 0,\ u''(0) = -1,\ u(x) = \cos x$$

[TYPO: The integrand should either be $u(t)$ or $(\frac{\pi}{2} - t)u(t)$. The latter will be used in the solution.]

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### Solution

Substitute the function in question on both sides of the integro-differential equation.

$$\frac{d^3}{dx^3}(\cos x) \stackrel{?}{=} 1 + \sin x - \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - t\right) \cos t\,dt$$

$$\sin x \stackrel{?}{=} 1 + \sin x - \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - t\right) \cos t\,dt$$

Use integration by parts to solve the integral. Let

$$v = \frac{\pi}{2} - t \qquad dw = \cos t\,dt$$

$$dv = -dt \qquad w = \sin t$$

and use the formula $\int v\,dw = vw - \int w\,dv$.

$$\sin \stackrel{?}{=} 1 + \sin x - \left[ \left(\frac{\pi}{2} - t\right) \sin t\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin t(-dt)\right]$$

$$\stackrel{?}{=} 1 + \sin x - \left[\left(\frac{\pi}{2} - \frac{\pi}{2}\right) + \int_0^{\pi/2} \sin t\,dt\right]$$

$$\stackrel{?}{=} 1 + \sin x - \left[(-\cos t)\Big|_0^{\pi/2}\right]$$

$$\stackrel{?}{=} 1 + \sin x - (0 + 1)$$

$$= \sin x$$

Therefore,

$$u(x) = \sin x$$

is a solution of the Fredholm integro-differential equation.